THEORY OF NUMBERS AND OF EQUATIONS.

Introduction a l'Étude de la Théorie des Nombres et de l'Algèbre Supérieure. Par ÉMILE BOREL et JULES DRACH, d'apres des Conférences faites a l'École Normale Supérieure, par M. JULES TANNERY. Pp. ix.+350. Paris, Nony, 1895.

THE beginning of the present century was signalized by epoch-making advances in two of the most interesting and beautiful fields of mathematics. Gauss crystallized the theory of numbers into a well-formed and polished science in that masterpiece of mathematical genius, the *Disquisitiones Arithmeticæ*, while the acumen of the brilliant Galois pierced the mist which had enveloped the theory of algebraic equations, and gave to the mathematical world the clue to the path which must be followed in their further treatment. The work of these masters has borne rich fruit, and we have now reached the stage at which the researches so inspired, have been carried far enough to admit of their results being collated from the journals and other scattered places of original publication, reduced to a common notation, unified, sometimes simplified, and the whole presented in an orderly systematic form. The need of such work has been felt less in the theory of numbers because the treatment of Gauss was so thorough and complete, his results so clearly put and so well arranged that the *Disquisitiones Arithmeticæ* might well serve a beginner as introduction to the subject, and that along the principal lines worked out by Gauss, there was left possible for subsequent writers little more than to paraphrase and illustrate the results which he had reached, to carry out the suggestions which he had not elaborated, and to extend his methods into new fields wherever he had left this possible. We have for some time, therefore, been in possession of several works on the theory of numbers which are more or less close paraphrases of Gauss, and which give an excellent survey of the subject, and in addition, there have been begun within the past few years two treatises (those of Bachmann and Mathews) which set themselves the task, to outline, in connection with a presentation of Gauss's theory, what has been done since Gauss. In the theory of algebraic equations, on the other hand, the great abstractness of the subject itself, heightened by the form of presentation adopted by Galois and by some other workers in this field, notably Kronecker, have combined to make an introductory treatise presenting the theory of equations as it has taken

shape during the present century, sketching clearly the fundamental principles and the most important results, a great desideratum. It is to be hoped that the forthcoming elaboration of Kronecker's lectures on the theory of algebraic equations, and likewise Weber's "Lehrbuch der Algebra," of which the first volume has but recently appeared, will do much to meet this need.* The work before us may also be ranked as a noteworthy step in this direction. It was inspired by the lectures of Mr. Jules Tannery, given during the academic year 1891–92, on the subjects of Arithmetic and Algebra. Two of his pupils undertook to edit and publish their notes of his lectures. In carrying out this undertaking, the lectures were so completely recast, that the work appeared under the title given above, with the names of Messrs. Borel and Drach upon the title-page as authors. Mr. Borel prepared the portion relating to the theory of numbers, while Mr. Drach prepared that on algebra.

The task of the reviewer is chiefly to sketch rapidly the contents of the work, so as to give the reader some idea of its nature and scope, and subsidiarily, to indicate the manner in which the work has been executed and to what reader it may prove most useful. We shall endeavor to do these things in turn.

PART I.—THE THEORY OF NUMBERS. Pages 3–156.

CHAPTER I: *General Properties of Congruences.* This chapter discusses in condensed form the leading elementary properties of congruences, Fermat's theorem and Wilson's theorem, with their generalizations and the properties of the function $\varphi(m)$. Of the subjects taken up, the last named is treated the most extensively, and in that connection is given a neat proof, due to Kronecker, of the following formula due to Euler:

$$\left[\sum_{n=1}^{\infty} \frac{1}{n^z}\right] \left[\prod_{(p)} \left(1-\frac{1}{p^z}\right)\right] = 1.$$

where p is an integer taking all prime values and z is any number whose real part is greater than unity. No reference is given to Kronecker's original publication of this proof.

CHAPTER II: *On Congruences with Prime Moduli.* The anal-

*Since the above was written the second volume has also appeared.

ogy between the theory of congruences and the theory of algebraic equations is a very evident one, and the latter subject can thus furnish a guide to the method and order of treatment of the former, and in the first study of the theory of congruences, which has almost always been preceded by some study of the general theory of algebraic equations, it is interesting and suggestive to make prominent the points of similarity and of difference between the two subjects. The work before us is the first one to point out and explicitly to emphasize and utilize this resemblance. This is a point of considerable merit which conduces both to clearness and to brevity in the demonstrations, since it often suffices to indicate the general methods and to omit details except in points of divergence from the corresponding algebraic proofs. The theory of roots of congruences as usually treated would find its precise counterpart in the theory of algebraic equations if in the latter only rational values were admitted. It is only when irrational and complex quantities are admissible as roots, that the fundamental theorem that every algebraic equation has a root can be established. The introduction of the so-called "Galois-imaginaries" allows an analogous generalization to be made in the theory of congruences, and a brief account of the Galois-imaginaries is therefore given. Though succinct and confined to the first properties, the account is clear and serves well as an introduction to the fuller treatment of these quantities given elsewhere. As our authors give no detailed reference either to original sources or to other presentations, it may not be amiss to add here that several such references may be found in Klein-Fricke, Elliptische Modulfunktionen, vol. I., p. 420.

CHAPTER III: *On Binomial Congruences.* Primitive roots and indices are first treated. Only the case of prime moduli is taken up, and for it the principal properties of primitive roots and indices are developed in the ordinary way. Then follows an extension of the theory to Galois-imaginaries, and then, returning to real numbers, the consideration of the congruence

$$x^n \equiv D \pmod{p}$$

is taken up in quite a little detail, after which the facts concerning primitive roots of composite moduli are briefly stated (in two pages).

CHAPTER IV: *Quadratic Residues; The Law of Reciprocity.* A first section discusses the number of solutions of

$$x^2 \equiv D \pmod{m}$$

for the various forms of m, and thereupon follow the usual definitions and proofs regarding quadratic residues. Jacobi's generalization of Legendre's symbol is given, without alluding to Jacobi, however, and some of its properties deduced according to Kronecker. The notation is misleading, in that p, which previously had denoted a prime, and usually does so, now denotes a composite number. This is the more misleading in that the property

$$\left(\frac{N}{P}\right)\left(\frac{N'}{P}\right) = \left(\frac{NN'}{P}\right)$$

is proved, while the other property

$$\left(\frac{N}{P}\right)\left(\frac{N}{Q}\right) = \left(\frac{N}{PQ}\right)$$

is not even mentioned.

The Gaussian Lemma is developed, and the Law of Reciprocity is proved according to a proof of Kronecker. As usual, our author gives not the slightest clue to aid one who wishes to find the proof as published by Kronecker himself. Slight allusion is made to the host of other proofs of the Law of Reciprocity.

CHAPTER V: *Decomposition of Numbers into Squares.* The general theory of quadratic forms is not taken up, but the proposition that every integer can be expressed as the sum of four or fewer squares is considered, and besides a proof along the usual lines, a proof by Smith is given which is based on the properties of continued fractions and makes use of determinants. The curious reader who might wish to find out how Mr. Smith himself expressed this proof, and what other things he said in the same connection is not given even the initials of Mr. Smith's name, to aid in differentiating the author of the proof from the non-mathematical persons bearing the same surname. A few pages are now devoted to the consideration of complex integers of the ordinary type, $a+bi$, (quite distinct from the Galois imaginaries, of course), and the fundamental theorem is established that in order that a complex integer be prime, it is necessary and sufficient that its norm be a real prime. Four pages, largely taken up with definitions, are given up to quadratic forms, and therewith the part of the book relating to the theory of numbers is brought to a close.

PART II.—HIGHER ALGEBRA. Pages 157–336.

CHAPTER I: *Elementary Algebra.* The author does not take up the ordinary theory of algebraic equations as it is presented in the current works on that subject, but the modern theory of equations as founded by Gauss, Abel and especially Galois, is the subject of discussion.

The properties of positive integers are first considered and the associative, the commutative, and the distributive laws of combination of these quantities by addition and by multiplication are set forth. Negative integers are next defined and considered from the purely logical point of view, and their various properties established. The totality of positive integers form what is known technically as a "group," when the law of combination is addition as well as when it is multiplication. This fact furnishes occasion for the introduction at this point of the group-concept, and an interesting illustrative discussion of the two groups of numbers which have just been mentioned. These sections on positive and negative integers would make admirable reading for the teacher of elementary algebra who finds that the concept of *negative numbers* is difficult of grasp to the beginner. Of course, it is not meant that the strict logic of our author is suitable for exact reproduction to one who has never before heard of negative numbers, but it is a clear and succinct presentation of the theory of negative numbers, which would make, in the teacher's mind, the skeleton of the work.

Fractional numbers are introduced into the system by the same logical processes as negative numbers, and then we pass to polynomials in one variable, with rational coefficients; addition, multiplication and decomposition into rational linear factors and decomposition into rational factors generally are considered, bringing out strongly the analogy with the decomposition of integers into prime factors.

This chapter on elementary algebra is clearly, concisely and logically written, and well repays reading.

CHAPTER II: *Algebraic Numbers.* Algebraic numbers, *i. e.*, the roots of algebraic equations with rational coefficients, are now defined and added to the system of rational numbers already on hand, and their properties are investigated in a manner analogous to that in which rational numbers were treated in the preceding chapter. Thus several of the well-known laws of the decomposition of algebraic polynomials into factors are deduced, and the fundamental problem of the theory is formulated:

Considering the totality of algebraic numbers, to determine what ones it is necessary and sufficient to introduce into the calculus, that is to say, to represent by explicit symbols, in order that all other algebraic numbers can be rationally expressed in terms of these.

The nature of this problem is illustrated by an analogous case from the theory of rational numbers, the problem itself, however, is not considered here, but constitutes the nucleus of what follows, especially of the fourth chapter. Algebraic functions, the theorem of D'Alembert algebraically interpreted, and symmetric functions with their properties and modes of calculation, are each the subject of a short section. Considering an irreducible algebraic polynomial of degree n, and with integral coefficients, the idea of the adjunction of its root, $\xi_1, \xi_2, \cdots\cdots \xi_n$, to the totality of numbers heretofore in use, viz., the rational numbers, is now developed, and the properties of rational functions of the elements of this new "domain," and the question of reductibility therein are taken up. Defining algebraic *integers* to be the roots of irreducible algebraic equations with integers as coefficients, and with unity as the coefficient of the highest power of x, it appears readily that it is sufficient to consider in a given domain, only equations whose roots, so far as they lie in the domain, are integers of that domain.

CHAPTER III: *Systems of Equations.* Under the caption of linear equations, the theorem of Rouché concerning the solution of p equations linear in n variables $p \gtreqless n$, is concisely given, but without any reference to the place of publication or even to the author of the theorem. The resultant of two polynomials in one variable is taken up, the form of elimination known as Sylvester's is explained, the resulting determinant is written in a slightly different form from that which is ordinarily used, and a determinant criterion is deduced for the existence of a common divisor of order q as follows:

If R denote the resultant expressed in Sylvester's method as a determinant of order $m+n$, and if R_i denote the determinant of order $m+n-2i$ formed by striking out of R the i first and the i last rows and columns, then the necessary and sufficient condition that the polynomials have a common divisor of order q is

$$R = R_1 = R_2 = \cdots\cdots = R_{q-1} = 0.$$

Other methods of forming the resultant are not considered, but the principal properties of the resultant are briefly deduced. Systems of two equations in two variables are dis-

cussed at some length, and the general properties of their resolvent explained, with a sketch of the extension to the case of p equations in n variables.

CHAPTER IV: *The Calculus of Algebraic Integers.* This chapter treats further of the formation of an algebraic "domain" and of calculation therein, as begun in Chapter II., of the nature of the process of "adjunction" introduced by Galois, and some of the properties of the "Galois-resolvent" are clearly set forth. An important step is made in the solution of the fundamental problem announced in the second chapter. Defining as *normal equation* every irreducible equation with integral coefficients which has the property that all its roots are integral functions, with integral coefficients, of one of them, and defining as *normal* algebraic integer, every algebraic integer which satisfies a normal algebraic equation it is proved that:

Every algebraic integer is an integral function with integral coefficients of a normal algebraic integer.

The field of research in our fundamental problem, which began with the totality of all algebraic numbers, is now narrowed to that of all normal algebraic integers.

Special equations are next taken up, but only sufficiently to permit of the introduction of the idea of their "group," and this, in turn, calls for the definition of "substitutions" and a brief consideration of their laws of combination considered as abstract operations, and of isomorphism, thus preparing the way for fuller treatment in the next chapter.

CHAPTER V: *Groups of Substitutions.* In twenty-three pages this chapter treats briefly the effect on rational functions of all possible substitutions on n letters, and also the general properties of groups and sub-groups and of Lagrange's resolvent.

CHAPTER VI: *Composite Groups.* This chapter takes up briefly in turn the series of composition of composite groups, the criterion of Galois for the solvability of algebraic equations, the proof that equations of degree greater than four are not algebraically solvable, and closes with a few words concerning the icosahedron group.

CHAPTER VII: *Applications; Conclusion.* This chapter treats of normal equations, and here the last step is made in the treatment of the fundamental problem by showing that, if we define as a *simple* equation an equation whose group is simple:

Every normal algebraic integer can be obtained by successively introducing into the calculus normal algebraic integers which are roots of simple normal equations.

The variety of normal equations known as Abelian equations, receives brief separate treatment and a few concluding remarks, followed by a note on Fermat's theorem, and one on the Galois imaginaries, close the volume.

This work may be characterized either as elementary or as advanced, if the sense in which the terms are used is explained, but either predicate would be misleading if used alone without explanation. The work is perhaps best described as "elementary advanced." It has been indicated at various points in the preceding sketch of the contents of the work, that it touches upon recent researches in the subjects under consideration, researches of which a first presentation could not take cognizance with profit to the beginner; in this sense the work is *advanced;* but only the most general outlines of the advanced topics are given; in this sense the work is *elementary.* It is not a work for beginners, though no specific acquisitions in these subjects seem to be presupposed. The part on the theory of numbers, while to some extent included in the preceding characterization, may with propriety be called elementary, and indeed might well be read by a beginner, but it does not cover a field sufficiently wide to make an adequate introduction to the subject. The part on algebra is an elementary introduction to the advanced region of the subject, and presupposes considerable training in abstract mathematical reasoning. There are very few illustrative examples and no exercises whatever, to which the reader may apply for himself the principles which have been set forth. This fact alone, even if the ground covered were quite appropriate to a beginner, would to the mind of the reviewer, make the book seem unsuited for a first course. The value of the work lies in its clear, concise outlines of general principles stripped of illustrations and amplification; such a presentation, even though it technically presupposes nothing, yet is of full value only to him who is quite familiar with the details which are suppressed, and with the point of view from which they are ordinarily seen. The general trend of the work has been well indicated by Mr. Tannery in the first paragraph of his Preface:

"During the scholastic year 1891–2, I gave some lectures on Arithmetic and Algebra to the students of the third year in the Normal School. I had in no sense the purpose of setting forth dogmatically portions of the science which have, indeed, always aroused a lively interest in me, but which I have not studied in detail; I wish solely to direct the curiosity of my hearers towards problems which are

among the most beautiful of those which mathematics presents, and towards methods which have been found or perfected by men of a genuis singularly rare and penetrating. The incessant advance of Analysis and of Geometry should cause neither these problems nor these methods to be forgotten. It is well enough known besides, that the different branches of mathematics are not independent and that those properties of integers which are the subjects of study in Arithmetic and Algebra, appear also in many questions arising in Analysis. Finally, it must be remembered that the teaching of the most elementary parts of Arithmetic and Algebra, of those parts which are taught in the *lycées* presupposes that the professor, if he wishes really to dominate his subject, has had at least some glimpses of the more advanced part of the science. My intention was simply to induce my hearers to look in this direction."

To say that this end has been well attained, that the clear logic, the well ordered arrangement, the attractive style, the keen analysis of fundamental principles combine to make this work an interesting introduction to the borderland of algebra such as Mr. Tannery purposed to make it, is to give the work but its due meed of praise, and leaves little more to be said. It is not a book of reference, it is a book to be read, and the scanty table of contents offers but little assistance to one, who, without having read the book, or at least having familiarized himself with the details as to its contents, wishes to consult its pages on a specific question.

The work is marred by one serious blemish. Almost no references are given either to the original sources of the material used, or as guides to those who wish to study the subject further. The meager references which are given are usually of the most elusive character, either simply a name, or the title of a large work, or of a journal without year or volume. This lack of references, which has been specifically indicated in passing at a number of points in the preceding sketch of the contents of the work, would be a grave defect in any work, but it is doubly so in a book like that before us which simply present the broad outlines and prominent features of a vast subject, gives fundamental principles, but few details, and serves chiefly to arouse interest and to introduce to a fuller study of the questions which have been raised. In such a work references and bibliography should be especially copious and detailed.

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QUATERNIONS.

A Primer of Quaternions. By ARTHUR S. HATHAWAY. New York, The Macmillan Co., 1896. 8vo. x+113 pp. 90c.

In this compact little book the author has compressed much of the theory of quaternions. The treatment is strictly Hamiltonian in method though a little different in form. The author seems to have caught the spirit of the master admirably and as a result the real simplicity and beauty of quaternions breathes from every page. The author rightly lays stress upon the *number* (*i. e.*, *multiple number*) idea of a quaternion. The term *vector* is not used to indicate a *directed length*, but to indicate a *right quaternion*. As a right quaternion a vector must possess an axis and this axis may be represented by a direction, and thus a definite length in a definite direction (*a step*) may *represent* a vector, but it *is* a step and not a vector. "In none of these cases is the concrete quantity an absolute number." This discrimination between step and vector (although the term vector is used differently) is Hamiltonian. It is a move in the right direction, away from the too geometrical and too physical view that has unfortunately always pervaded treatises on quaternions. It does not seem entirely the wisest way to present a subject *through* its applications, for in this way much fog and much recrimination has been brought about. Whatever we may say, we must at last admit that quaternions is an *algebra* and as such has *many* applications, the usual one being only one, which form really no inherent part of the subject *in abstracto*. Many so-called modifications of it are easily perceived to be from this point of view *other algebras*. The point the author insists upon rightly is that a quaternion is a further extension of the idea of number than that of the ordinary complex number, but similar. No one ever confuses the *representation* of a number $a+b\sqrt{-1}$ in the complex plane with the number itself. So a vector and its representation are two different things.

Chapter I. deals with the *step* and its use. We find the usual kind of examples and theorems of what is generally called vector analysis. The examples are of the kind best calculated to stimulate a student—they are suggestive of further developments to him and if he has any originality cannot fail to develop it. Such examples however are not strictly quaternionic.

Chapter II. discusses the rotations of steps and leads up to multiple numbers which are really quaternions, such as

(2, 30°). We meet here the old familiar term of Oliver, Wait, and Jones's algebra—*versitensor*. The use of arcs to represent rotations is developed and the student is set a few examples on spherical conics which are to be used in the following chapter.

Chapter III. introduces the quaternion and the names of its various elements. There is no appeal to the four-unit form in proving the usual introductory theorems—in fact, Cartesians are abandoned entirely throughout the text, a phase which is again thoroughly Hamiltonian. The rotator $q(\ )q^{-1}$ is introduced; powers and roots of quaternions are considered; the products of vectors, conditions of perpendicularity and parallelity, the distributivity of quaternion multiplication and related subjects are handled with skill and clearness. Geometric applications appear here, and some development of trigonometry. The examples were, of course, meant to be illustrated and analyzed by the instructor. Few pupils with the knowledge to be had up to this point would be able to handle them well. They are of the suggestive kind, calculated rather to point a way than to furnish an exercise.

Chapter IV. is perhaps the most compact of all the book. Within a limit of thirty pages are discussed equations of the first degree, their applications to the line and plane, nonions or the linear vector operator, linear homogeneous strain, nullity and vacuity of a nonion, and applications to surfaces of the second order. Almost as much is brought out or hinted at as Tait devotes one hundred and twenty large pages to. Yet we can conceive that under proper teaching and not too hasty progress the work could be done. The work in nonions brings in many ideas which have come from matrices, though easily enough developed by quaternion processes. The chief one of these is the notion of nullity.

The author tells us the work has had the test of the classroom and that it may follow ordinary algebra and geometry, and that he has given it as a substitute for solid analytics. This surely certifies to the feasibility of the use of the text. It is to be hoped that it will find the success it deserves and be followed by its natural sequel.

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RECENT TEXT-BOOKS OF GEOMETRY.

Elements of Geometry. By GEORGE C. EDWARDS, Associate Professor of Mathematics in the University of California. New York, The Macmillan Company, 1895. 8vo., pp. xvi + 293.

Plane and Solid Geometry. By WOOSTER WOODRUFF BEMAN, Professor of Mathematics in the University of Michigan, and DAVID EUGENE SMITH, Professor of Mathematics in the Michigan State Normal School. Boston, Ginn & Co., 1895. 8vo., pp. ix + 320.

Plane and Solid Geometry. Suggestive Method. By C. A. VAN VELZER, Professor of Mathematics in the University of Wisconsin, and GEO. C. SHUTTS, Professor of Mathematics in the Wisconsin State Normal School. Madison, Wis., Tracy, Gibbs & Co., 1894. 8vo., pp. viii + 395.

These geometries are written by authors whose attainments prepare them for careful and thorough work. The author of the first is associate professor of mathematics in the University of California, and the remaining authors are professors of mathematics in the Universities and Normal Schools of the States of Michigan and Wisconsin, respectively. This combination of joint authorship in each of these States shows that the two great schools of each State work in harmony with each other.

1. Professor Edwards has followed an unfortunate plan of logical development which mars an otherwise excellent and original work. Several popular text-books in the United States employ the directional method of defining a straight line, and apparently get rid of or deduce Euclid's axiom of parallels from it just as Professor Edwards has done. The authors of these books do not seem to be aware of the startling position in which they have thus placed themselves. The directional definition accomplishes no more for the straight line than Euclid's or others that have been employed, and simply makes straight lines such lines that, if two of them are placed with any two points in coincidence, they will coincide throughout. A hemispherical surface furnishes a moderately good analogue of a non-Euclidean plane. Suppose the physical conformation of the light medium were such that rays of light traveled in arcs of great circles on this surface and were absorbed at the boundary. Then, according to the directional definition, arcs of great circles would be straight lines, and all that can be logically derived from the definition holds for this spher-

ical surface. In particular, Professor Edwards's proof that the sum of the exterior angles of a triangle is four right angles, holds word for word for spherical triangles, and hence proves that the sum of the angles of any spherical triangle is exactly two right angles. According to the Autocrat of the Breakfast Table, "logic is logic," and the authors of the directional method have given us a new phase of it equal in every respect to the logic of the wonderful "one-hoss-shay."

Professor Edwards's book contains many noteworthy features. What may be called the descriptive method of proof that is common in higher mathematics is adopted throughout the work, all formal proofs being discarded. This method has many advantages. It offers opportunities for bringing out the leading lines of thought and methods of attack, and of making clear to the student those parts that are most important and that require greatest emphasis. The examples, explanations and illustrations are excellent. The error in logical development will, in the hands of a good teacher, prove of educational value in bringing out the points I have named, and will serve to forcibly illustrate the exceeding delicacy of logical reasoning and the care that every student must exercise in avoiding whatever seems plausible until assured that it has a sound logical basis.

2. Professors Beman and Smith have written a geometry that has many merits. The plan of development is essentially that laid down by the Association for the Improvement of Geometrical Teaching, of England. The book introduces many modern ideas. Thus, the principle of duality is introduced in connection with the first propositions of Book I., and the student is shown how to use it in pairing his propositions and in obtaining suggestions of new propositions. The principle of symmetry is also introduced early, and central and axial symmetry are shown to be dually related. The principles of continuity and of positive and negative relations are also explained in the first book. These are all important and modern principles of geometry. The properties of similar figures are presented in such a way as to lead naturally into the properties of perspective. The logic is generally good and clear. The construction of formal proofs is particularly emphasized, and large lists of examples are furnished for original exercise. The student who uses this book will have better conceptions of modern geometrical methods and principles than can be obtained from the usual text-books. In the writer's opinion elementary text-books should go even far-

ther than the authors have done and not stop short of enabling the student to handle the compass and rule in the modern way.

3. The geometry of Professors Van Velzer and Shutts is characterized by their phrase, "suggestive method." In general style and arrangement it is very much after the pattern of Chauvenet's geometry. Although the authors give credit in this respect to the revised edition of Chauvenet, they have very properly avoided the directional method which renders that revision considerably inferior in point of logic to the old edition. This book seems to be an excellent successor of that much-valued treatise. The important innovation of the suggestive method consists in giving, in connection with each proposition, suggestions as to proof, which the student is expected to expand into a formal proof after the pattern of a few "models" that are given in each book. The idea is an excellent one, and cannot fail to develop geometrical insight and originality. The regular sequence of propositions is supplemented by lists of original exercises.

It will be seen that each of the above text-books embodies new and distinctive features. The first and second make use of algebraic analysis much more extensively than has hitherto been common in works on elementary geometry. With the exception of the errors due to the use of the directional method by Professor Edwards, they are all superior to text-books that are in common use, in which numerous errors of this kind are often contained. The logical development of the foundations of a subject is a very delicate matter and requires the most careful and diligent thought of able minds to clearly elucidate it. Elementary writers and students cannot, therefore, always understand the necessity for this or that logical arrangement, since the amount of work and study that has been put upon it does not show upon the surface. A correct line of demonstration may be extremely difficult to elementary students because the whys and wherefores of it extend far beyond their depth, but this furnishes no excuse for presenting them with a false line of demonstration whose plausibility is less exacting upon their mental powers.

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ON SEVERAL THEOREMS OF OPERATION GROUPS.

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§ 1.

In a recent number of the *Quarterly Journal of Mathematics* (vol. 28, p. 233) we proved the theorem, "Every group (G) whose order is divisible by p^4, p being any prime number, contains a commutative group (G_1) of order p^3." The following proof of this theorem is much simpler and can readily be extended to apply to more general theorems.

G contains a subgroup (G') of order p^α, $\alpha > 3$. G' contains a subgroup of order p whose substitutions* are commutative to all the substitutions of G'.† With respect to this subgroup G' is isomorphic to a group (G_1') of order $p^{\alpha-1}$. G_1' contains a subgroup of order p whose substitutions are commutative to all the substitutions of G_1'. With respect to this subgroup G_1' is isomorphic to a group (G_2') of order $p^{\alpha-2}$. Hence we may suppose the substitutions of G' so arranged that the first $p^\beta (\beta = 0, 1, 2, 3, \cdots, \alpha - 1)$ constitute a self-conjugate (invariant) subgroup of G' and that each of its p sets of $p^{\beta-1}$ substitutions, in order, is transformed into itself by all the substitutions of G'.‡

If we suppose $\beta = 2$ each of the p sets contain p substitutions. The substitutions of $p - 1$ of these sets must be transformed, by all the substitutions of G', according to the cyclical group of order p or according to identity. Those in the first set are known to be transformed according to identity. Hence each of these p^2 substitutions must be commutative to at least $p^{\alpha-1}$ substitutions of G' and the the first p^4 in the given arrangement must contain a commutative group of order p^3. This proves the given theorem.

In general, the first $p^{\beta-1}$ substitutions in the given arrangement are transformed by G' according to a group (H) of order p^θ. A substitution which is commutative to all the substitutions in the second set of $p^{\beta-2}$ is commutative to each of the given $p^{\beta-1}$ substitutions. Hence H is simply isomorphic to a group whose degree cannot exceed $p^{\beta-2}$ and the maximum value (M) of θ is given by the formula §

* The operations are throughout represented by means of substitutions.
† SYLOW, *Mathematische Annalen*, vol. 5, p. 588.
‡ Ibid.
§ Cf. DIRICHLET-DEDEKIND, Zahlentheorie, p. 27.

$$M = p^{\beta-3} + p^{\beta-4} + p^{\beta-5} + \cdots + 1.$$

This proves the

THEOREM I: *The substitutions of a group of order p^α, where $\alpha > p^{\beta-3} + p^{\beta-4} + \cdots + p + 1 + (\beta - 1)$, can be so arranged that the first $p^{\beta-1}$ constitute a self-conjugate subgroup whose substitutions are commutative to the substitutions of a subgroup of order p^β.*

By making $\beta = 3$ in this theorem we readily obtain the former theorem, since the groups of order p^2 are commutative. Another important extension of the given theorem is based upon the fact that the given commutative group of order p^3 is a *self-conjugate* subgroup of G'. The proof of this fact follows almost directly from the given arrangement. It is only necessary to consider the case when the first p^3 substitutions in this arrangement are not commutative. Since the second set of p substitutions must then be transformed according to the transitive group of order p they are commutative to only p^3 of the first p^4 substitutions. Hence there can be only one commutative group of order p^3, among the first p^4 substitutions, that contains the first p^2 substitutions. Since conjugate subgroups have the same properties this must be self-conjugate.

Sylow's proof of the theorem that every group of order p^α contains at least p substitutions which are commutative to all the substitutions of the group applies equally to all its self-conjugate subgroups.* We may therefore use any self-conjugate subgroup of order p^β for the first p^β substitutions in the given arrangement of the substitutions of G'. Hence the properties which we have observed in regard to the first p^β substitutions belong to each of the self-conjugate subgroups of this order.

In particular, every self-conjugate subgroup of order p is composed of substitutions which are commutative to the entire group.† This is also a particular case of the theorem that a group of the order p^α cannot transform any of its substitutions of order p into its α power unless $\alpha = 1$. Since p has primitive roots, the $p - 1$ power of the transforming substitution would have to be commutative to the transformed substitution. From this it follows that the first power of the transforming substitution would also have to be commutative to the transformed substitution.

As we may consider the first p^3 substitutions commuta-

* Cf. FROBENIUS, *Sitzungsberichte der Berliner Akademie*, 1895, p. 983.
† Ibid, p. 982.

tive, in the given arrangement of all the substitutions of G', we readily obtain from theorem I. the following

Corollary. Every group of order p^α, $\alpha > p+4$, contains a commutative group of order p^4.

In particular, for $p=2$, we have that every group whose order is a power of 2 and larger than 64 contains a commutative group of order 16. In the given article we proved that there are seven groups of order 32 that do not contain such a commutative group. We shall presently construct a group of order 64 which contains no commutative group whose order exceeds eight. This will prove that we cannot assign a smaller value to α in the given corollary, at least not without restricting the values of p.

Suppose that the following group (G') of order 32 is made simply isomorphic to itself in such a way that the elements in the corresponding substitutions differ only with respect to subscripts.

1*

ae. bf. cg. dh. im. jn. ko. lp. qu. rv. sw. tx. yγ. zδ. aε. βθ
ai. bn. ck. dp. fj. em. hl. go. qy. rδ. sa. tθ. vz. uγ. xβ. wε
am. bj. co. dl. ei. fn. gk. hp. qγ. rz. sε. tβ. uy. vδ. wa. xθ
aceg. bdfh. ikmo. jlnp. qsuw. rtvx. yaγε. zβδθ
agec. bhfd. iomk. jpnl. qwus. rxvt. yεγa. zθδβ
akeo. bpfl. cmgi. djhn. gaue. rθvz. sγwy. tzxδ
avek. blfp. cigm. dnhj. qεua. rβvθ. sywγ. tδxz
abcdefgh. ijklmnop. qrstuvwx. yzaβγδεθ
adgbehcf. ilojmpkn. qtwruxsv. yjεgγθaδ
afchebgd. inkpmjol. qvεxurwt. yδaθγzεβ
ahgfedcb. iponmlkj. qxwvutsr. yθεδγβaz
ajgpencl. bohmfkdi. qzwθuδsβ. rεxγvαty
alcnepgj. bidkfmho. qβsδuθwz. rytavγxε
anglejcp. bkhifodm. qδwβuzsθ. raxyvεtγ
apcjelgn. bmdofihk. qθszuβwδ. rγtεvyxa
ag. bz. cw. dθ. eu. fδ. gs. hβ. iy. jr. kε. lx. mγ. nv. oa. pt.
au. bδ. cs. dβ. eq. fz. gw. hθ. iγ. jv. ka. lt. my. nr. oε. px.
ay. bv. cz. dt. eγ. fr. ga. hx. iq. jδ. kw. lβ. mu. nz. os. pθ.
aγ. br. ca. dx. ey. fv. gε. ht. iu. jz. ks. lθ. mq. nδ. ow. pβ.
as. bβ. cq. dz. ew. fθ. gu. hδ. ia. jt. ky. lr. mε. nx. oγ. pv.
aw. bθ. cu. dδ. es. fβ. gq. hz. iε. jx. kγ. lv. ma. nt. oy. pr.
aa. bx. cy. dv. εs. ft. gγ. hr. is. jθ. kq. lδ. mw. nβ. ou. pz.
aε. bt. cγ. dr. ea. fx. gy. hv. iw. jβ. ku. lz. ms. nθ. oq. pδ.

* It may be observed that these substitutions are arranged in the given manner, *i. e.*, the first p^β ($p=2$, $\beta=0, 1, 2, 3, 4$) constitute a self-conjugate subgroup of G' and the first p^3 are commutative. That each of the first p sets of $p^{\beta-1}$ substitutions is commutative to G' gives additional information only when $p>2$.

arkθevoβ. bαpufεlq. cxmδgtiz. dyjshγnw.
atkzexoδ. bγpwfyls. crmθgviβ. dajuhεnq.
avkβeroθ. bεpqfalu. ctmzgxiδ. dγjwhyns.
axkδetoz. bypsfrlw. cvmβgriθ. dεjqhanu.
azoteδkx. bwlγfεpy. cθirgβmv. dunahqjε.
aβoveθkr. bqlεfupα. cżitgδmx. dwnγhsjy.
aδoxezkt. bslyfwpγ. cζivgθmr. dqnεhuja.
aθoreζkv. bulαfqpε. cδixgzmt. dsnyhwjγ.

If we multiply the resulting group (K) of order 32 and degree 64 by

$brh\beta.\ ckgo.\ d\theta fv.\ im.\ j\delta px.\ ltnz.\ qs\gamma a.\ uwy\varepsilon.\ b_1 r_1 h_1 \beta_1.\ c_1 k_1 g_1 o_1.$
$d_1 \theta_1 f_1 v_1.\ i_1 m_1.\ j_1 \delta_1 p_1 x_1.\ l_1 t_1 n_1 z_1.\ q_1 s_1 \gamma_1 a_1.\ u_1 w_1 y_1 \varepsilon.\ t,$

where t is of the second order and interchanges the elements of K which differ only with respect to the subscript, we obtain the required group of order 64. For if this group should contain a commutative group of order 16, K would have to contain just 8 of its substitutions.* At least 4 of these would have to be found among the first 16 substitutions, since these constitute a self-conjugate subgroup of K. Hence we see that the tail to K does not contain any substitution which is commutative to the substitutions of a commutative group of order 8 found in K. The entire group can therefore not contain a commutative group whose order exceeds 8.

From these considerations it follows that while we can always assume the presence of a commutative group of order 8 in a group whose order is divisible by 16, yet we cannot assume the existence of a commutative group of order 16 unless the order of the group is divisible by 128. These facts are of practical importance in group construction since the presence of a large commutative group often simplifies the work very much.

§ 2.

THEOREM II: *If a group contains a subgroup whose order is equal to the order of the group divided by p (p being any prime number), and if this subgroup contains substitutions of order p^α, $\alpha > 0$, all the substitutions of this order for any given value of α, which are contained in the subgroup must generate a self-conjugate subgroup.*

Let the substitutions of the given subgroup (G) be

$$1, s_2, s_3, \cdots, s_i$$

If G is not self-conjugate, multiply it into any substitution

* *Quarterly Journal of Mathematics*, vol. 28, p. 250.

(t) which is not commutative to it. If the transform (G_1) of G with respect to any one of these products contains a substitution $(t^{-1}st)$ of order p^a which is not found in G and if s does not transform G_1 into itself we may form the following rectangle with p *conjugate* rows of l elements:

$$\begin{array}{cccc} t & s_2t & s_3t & \cdots s_lt \\ ts & s_2ts & s_3ts & \cdots s_lts \\ ts^2 & s_2ts^2 & s_3ts^2 & \cdots s_lts^2 \\ ts^{p-1} & s_2ts^{p-1} & s_3ts^{p-1} & \cdots s_lts^{p-1} \end{array}$$

All the substitutions of a given row transform G into the same group while any two substitutions from different rows transform G into two different groups. As no substitutions in the given rectangle can be equal to each other the entire group would have to contain at least $l(p+1)$ substitutions. This is contrary to the hypothesis. As the opposite to the assumptions made above would in each case assume the truth of the theorem, the proof is complete.

Corollary. If G is generated by substitutions of order p^a it is self-conjugate.

In particular, if we assume that the order of G is p^{a-1} we obtain on important theorem due to Frobenius, which he has proved in a number of different ways as a special case of other general theorems.*

§ 3.

By means of the operation $sts^{-1}t^{-1}$ we may readily determine the smallest self-conjugate subgroup with respect to which a group is isomorphic to a commutative group, or, in other words, the order of the largest commutative group to which a given group can be made isomorphic.† Since s and s^{-1} have the same sign and a group can be isomorphic to only one commutative group of the maximum order we have the following:

THEOREM III: *If we make a group isomorphic to the largest possible commutative group the substitutions which correspond to identity in the commutative group are positive.*

The application of the symbol $sts^{-1}t^{-1}$ to operation groups has a two-fold importance as it simplifies the study of a group with respect to its solvability and tends to bring into prominence some of the similarities which exist between Lie's transformation groups and the older operation groups

* *Cf. Sitzungsberichte der Berliner Akademie*, 1895, p. 962.

† *Quarterly Journal of Mathematics*, vol. 28, pp. 266–268.

which have been developed mainly under the form of substitution groups.

It may be well to add that the symbol $sts^{-1}t^{-1}$ has been used in substitution groups for a long time, but its use has been very limited. As far as we know its practical application to determine important properties of a group was first explained in the recent article in the *Quarterly Journal* to which we referred above.

GÖTTINGEN,
September, 1896.

NUMERICALLY REGULAR RETICULATIONS UPON SURFACES OF DEFICIENCY HIGHER THAN 1.

BY PROFESSOR HENRY S. WHITE.

By the term *reticulation* I shall designate for present purposes any system of lines lying upon a closed surface, together with all the points in which these lines intersect one another. Further I shall assume that they divide the surface into portions, of which each by itself is simply connected, *i. e.*, has deficiency zero. These portions of the closed surface may be termed *faces*, and their intersection points *vertices*, while each boundary line terminated by two consecutive vertices is an *edge*. If F, V and E denote the numbers of faces, vertices and edges, respectively, in a reticulation, and p the deficiency of the supporting surface, then Euler's relation for convex polyedra, generalized, will be

$$E = V + F + 2p - 2.$$

A reticulation is clearly entitled to be called *numerically regular* when it has:

1. In every vertex a constant number of termini of edges; call this number $\rho + 2 = r$.

2. In every circuit bounding a face a constant number of edges, call this number $\sigma + 2 = s$.

We may for the present regard these two numbers ρ and σ alone as characteristics of a regular reticulation; there will remain for subsequent inquiry the determination of the number of essentially different types having any given set of characteristics ρ, σ, and p. From these three the values of F, V, and E can be computed, as will be seen below. Counting then as one class all regular reticulations characterized by the same values of ρ and σ, it can be shown that *on a surface of given deficiency p, there can exist only a finite number of classes of numerically regular reticulations.*

To prove this we notice that the edges may be enumerated in two ways, either by counting the vertices or by counting the faces:

$$2E = Vr = V(\rho + 2),$$
$$2E = Fs = F(\sigma + 2).$$

Combining these equations with Euler's relation and eliminating two of the three letters E, V, F, we find expressions for the values of these three quantities in terms of ρ, σ and p.

$$F = \frac{(4p-4)(\rho+2)}{\rho\sigma-4}, \qquad V = \frac{(4p-4)(\sigma+2)}{\rho\sigma-4},$$

$$E = \frac{(2p-2)(\rho+2)(\sigma+2)}{\rho\sigma-4}.$$

The restriction of p to values greater than unity makes all three numerators positive integers. The three cases where the denominator is zero are common to all deficiencies, being the division of a surface into triangles, quadrilaterals and hexagons respectively; E, F and V being then infinite except for $p = 1$, which particular value makes them indeterminate. Omitting these cases, we see that positive values of F, V and E require positive values of $\rho\sigma - 4$. Accordingly $\rho\sigma$ has the lower limit 5, and we seek next to find an upper limit.

Two cases may be distinguished. Either $\rho = \sigma$, or else they are unequal. Since the formulæ for V and F are merely exchanged when ρ and σ are permuted, we shall assume in the second case that ρ is the greater.

From $\rho = \sigma$ it follows that

$$F = V = \frac{4p-4}{\sigma-2}$$

i. e., $\sigma - 2$ can have only those values that are factors of $4p-4$, a finite number, and the upper limit of $\rho\sigma$ is for this case $(4p-2)^2$. From $\rho > \sigma$ it follows also that $\rho\sigma < (4p-2)^2$. For since $\frac{V}{F} = \frac{\sigma+2}{\rho+2}$, we have $F > V$ and

$$F - V = \frac{(4p-4)(\rho-\sigma)}{\rho\sigma-4} \geqq 1,$$

$$(4p-4)(\rho-\sigma) - \rho\sigma \geqq -4,$$

adding $(4p-4)^2$ to each side, and factoring we have

$$(4p-4+\rho)(4p-4-\sigma) \geqq 4(2p-1)(2p-3).$$

Since $p>1$, the second member is positive, therefore

$$4p-4-\sigma>0.$$
$$\sigma<4p-4.$$

As $V \geqq 1$, $\rho\sigma - 4 \geqq (4p-4)(\sigma+2)$.
or $\rho\sigma < 4(4p^2-6p+3)$,
an upper limit for the second case, less than $(4p-2)^2$.

Since then $\rho\sigma$ has for every value of $p>1$ finite lower and upper limits, and since any admissible value has only a finite number of pairs of factors, and only those values of ρ and σ are to be retained which give integral values to V and F, it is now shown that for given p only a finite number of classes of regular reticulations can exist.

In tabulating the classes belonging to a given p, it is convenient to treat as one any two which differ only in that the values of ρ and σ, hence also of V and F, are permuted. In the table for $p=2$, we find 14 classes, representing 25 when such dual reticulations are distinguished.

REGULAR RETICULATIONS FOR $p=2$.

No.	$\rho\sigma$	ρ	σ	V_r	F_s	E
1	5	5	1	12_7	28_3	42
2	6	6	1	6_8	16_3	24
3	6	3	2	8_5	10_4	20
4	7	7	1	4_9	12_3	18
5	8	8	1	3_{10}	10_3	15
6	8	4	2	4_6	6_4	12
7	9	3	3	4_5	4_5	10
8	10	10	1	2_{12}	8_3	12
9	12	6	2	2_8	4_4	8
10	16	16	1	1_{18}	6_3	9
11	16	4	4	2_6	2_6	6
12	20	10	2	1_{12}	3_4	6
13	24	8	3	1_{10}	2_5	5
14	36	6	6	1_8	1_8	4

For realizing those number-schemes upon models, I have not formulated any *a priori* infallible method, but have treated the problem as a test exercise for the geometric im-

agination. My own success was limited to the production of two models. The other twelve, with one exception, were designed with great facility and neatness by Mr. OLIN H. BASQUIN, Fellow in Physics in Northwestern University. Any closed surface free from double lines and possessing visibly the deficiency 2 is suitable for displaying graphically these 14 (or 25) reticulations. Five have been illustrated upon small plaster forms cast for the purpose, double rings; and the entire series is traced upon flat cards through which two apertures have been made. Both sides of a card are considered as forming the surface, the opposite sides being connected by every apparent sharp edge. In only one instance did Mr. Basquin find it necessary to let the same line occur twice in the boundary of a single face; but often it is seen that the same point occurs twice or even thrice in the boundary of the same face. As an experimental fact it is worth while to record that *every class of numerically regular reticulations that is arithmetically possible upon a surface of deficiency $p = 2$ has been realized graphically in these 14 models, classes dual to each other being counted as one.*

The 14 regular reticulations of the above scheme for $p = 2$ are fundamental to certain others that exist on surfaces of every deficiency $p > 2$. From each of the former is derived one of the latter for each deficiency. Such a derivation is arithmetically evident, and Mr. Basquin has proposed a practical method for following out in the model the arithmetical indications. If I illustrate this method by derivation from $p = 2$ to $p = 3$, the extension of the process to $p = 4$, 5, etc., will be immediately evident; and the similar derivation from $p = 3$ to $p = 5$, 7, etc., or from $p = n + 1$ to $p = 2n + 1$, $3n + 1$, etc., will need no explanation.

If on a surface of deficiency 2 a regular reticulation is characterized by the numbers ρ', σ', V', F', E', then on a surface of deficiency 3 there is a regular reticulation *marked by the same* ρ', σ', together with values V, F, E, such that

$$V = 2V', \quad F = 2F', \quad E = 2E'.$$

For if ρ', σ' render V', F', E' integers, where

$$V' = \frac{4(\rho' + 2)}{\rho'\sigma' - 4}, \quad F' = \frac{4(\rho' + 2)}{\rho'\sigma' - 4}, \quad E' = \frac{2(\rho' + 2)(\sigma' + 2)}{\rho'\sigma' - 4},$$

then they render integral also the values V, F, E belonging to $p = 3$, $(4p - 4 = 8)$, where

$$V=\frac{8(\sigma'+2)}{\rho'\sigma'-4}=2V', \quad F=\frac{8(\rho'+2)}{\rho'\sigma'-4}=2F'$$

$$E=\frac{4(\rho'+2)(\sigma'+2)}{\rho'\sigma'-4}=2E'$$

All such sets of numbers V, F, E, obtained by doubling the numbers in the table for $p=2$, I shall call *derivative* sets for $p=3$; the corresponding reticulations will also be called derivative. All others I shall call *special* sets or reticulations for $p=3$. For $p=5$ there would be of course two kinds of derivatives; those coming from the scheme for $p=2$ by five-folding the numbers in the columns V, F, E, and those coming from the *special* sets for $p=3$ by doubling the members V, F, E. So for higher deficiencies there would be often several kinds of derivatives, but for a prime $(p-1)$ only one kind.

For $p=3$ the special regular reticulations are readily determined by trial; for the value of $(\rho\sigma-4)$ must contain at least the factor 8, with either ρ or σ an odd number, otherwise it will contain a factor 16.

SPECIAL REGULAR RETICULATIONS FOR $p=3$.

No.	$\rho\sigma$	ρ	σ	V_r	F_s	E
1	12	4	3	5_6	6_5	15
2	12	12	1	3_{14}	14_3	21
3	28	7	4	2_9	3_6	9
4	28	28	1	1_{30}	10_3	15
5	36	18	2	1_{20}	5_4	10
6	60	12	5	1_{14}	2_7	7
7	100	10	10	1_{12}	1_{12}	6

For these seven special classes Mr. Basquin has constructed card models. Here, therefore, we record, as an empirical fact, that *every special class of regular reticulations that is arithmetically possible upon a surface of deficiency 3 can be realized graphically.*

How to produce a derivative reticulation from its fundamental remains to be explained. With a model of some one reticulation of deficiency 2 before the eye, prepare or imagine a duplicate of it; and let both be of material easily cut and distorted. Cut the first along any closed line that does not divide it into separate parts. Make an exactly similar cut in the second model. Distort each severed ex-

tremity through 90° in such a direction that then the models can be opposed to each other and each severed portion can be united to the duplicate of that from which it was divided. The resulting surface will be of deficiency 3 and will contain a regular reticulation with twice as many vertices, faces and edges as its original. As illustrations of this mode of derivation Mr. Basquin has found fairly simple examples to arise from Nos. 3, 6, 9, 11 of the table for deficiency 2. It will be seen that the closed line marking the cut may be equally well any number, not greater than $p-1$, of non-intersecting closed curves. Where $p>2$, this observation points to an interesting variety of derivative reticulations.

Of the starred polyedra, one is found to belong to this scheme of regular reticulations, namely the starred dodecædron; in the others, the connectivity of the individual faces and vertices has to be taken into account. Semi-regular reticulations, analogous to the solids of Archimedes, I have not investigated; but I should expect that the graphical production of such would give the inventive faculties more exercise and pleasure than the construction of those wholly regular.*

NORTHWESTERN UNIVERSITY,
EVANSTON, ILL., *August*, 1896.

CORRECTION.

THE theorem given in the first paragraph of the article "Note on the Special Linear Homogeneous Group," p. 336, of the last volume of the BULLETIN, is not true, and is not, as there stated, a consequence of results given on p. 232. The theorem given in the second paragraph, p. 336, regarding the special linear homogeneous group in n variables, for $n=2$ or composite, and the method of proof which follows, holds also if n is an odd prime. HENRY TABER.

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*The theory of canonical dissections of a Riemann surface leads to the completion of the foregoing discussions.—H. S. W.

NOTES.

A REGULAR meeting of the AMERICAN MATHEMATICAL SOCIETY was held in New York on Saturday afternoon, November 28th, at three o'clock, the President, Dr. G. W. HILL, in the chair. There were twelve members present. On the recommendation of the Council, the following persons, nominated at previous meetings, were elected to membership: Mr. W. E. BROOK, University of Nebraska; Dr. L. E. DICKSON, University of Chicago; Dr. ARNOLD EMCH, University of Kansas; Dr. J. W. GLOVER, University of Michigan; Dr. J. I. HUTCHINSON, Cornell University; Lieut. H. H. LUDLOW, U. S. A., Jackson Barracks, New Orleans, La.; Mr. H. O. MURFEE, University of Virginia; Professor A. W. PHILLIPS, Yale University; Dr. VIRGIL SNYDER, Cornell University; Professor J. H. TANNER, Cornell University; Professor W. C. TINDALL, University of Missouri. Two nominations for membership were received.

The following paper was read:

Professor HENRY TABER: "Notes on the theory of bilinear forms."

AT the annual meeting of the London Mathematical Society, held on November 12th, the following officers and members of the Council were elected for the ensuing year: President, Professor E. B. ELLIOTT; Vice-Presidents, Major P. A. MACMAHON, MORGAN JENKINS and Dr. E. W. HOBSON; Treasurer, Dr. J. LARMOR; Secretaries, R. TUCKER and A. E. H. LOVE; other members of Council, Lieut.-Col. A. J. C. CUNNINGHAM, H. T. GERRANS, Dr. J. W. L. GLAISHER, Professor A. G. GREENHILL, Professor M. J. M. HILL, Professor W. H. H. HUDSON, A. B. KEMPE, F. S. MACAULAY and D. B. MAIR. Major MACMAHON, the retiring President, delivered an address, of which the subject was "The Combinatory Analysis." At this meeting the De Morgan medal was presented to Mr. SAMUEL ROBERTS, the fifth recipient of the medal.

UNIVERSITY OF MUNICH. The following mathematical courses are announced for the winter semester: By Professor BAUER: Theory of equations; Applications of the differential and integral calculus to curves and surfaces. —By Professor LINDEMANN: Analytical geometry of two dimensions; Theory of Abelian functions; Mathematical theory of life insurance.—By Professor PRINGSHEIM: Differential calculus; Fourier's series.—By Professor GRÖTZ:

Analytical mechanics.—By Dr. DÖHLEMANN: Descriptive geometry; Kinematics.—By Dr. ANDING: Theory of probabilities and method of least squares.—By Dr. WEBER: Introduction to analysis; Differential equations.—Dr. KORN: Calculus of variations with respect to its applications to physics.

THE scientific world suffers a serious loss in the death of Professor F. F. TISSERAND, who died of apoplexy at Paris, October 20th, aged fifty-one years. At first an assistant in the Paris Observatory, and later professor of astronomy and director of the observatory at Toulouse, he was called to Paris as professor of mechanics, afterwards becoming professor of astronomy. In 1892 he was made director of the Paris Observatory, succeeding the late Admiral Mouchez.

AT the celebration of the Princeton sesqui-centennial, October 22, 1896, the degree of LL.D. was conferred upon the following mathematicians: Professor J. W. GIBBS, Dr. G. W. HILL, Lord KELVIN, Professor FELIX KLEIN, Professor SIMON NEWCOMB, and Professor J. J. THOMSON.

THE number of honorary members of the New York Academy of Sciences was increased, Monday evening, October 5, by the election of Professor FELIX KLEIN, of Göttingen, and Professor J. J. THOMSON, of Cambridge, England.

DR. W. S. EICHELBERGER has resigned his position as instructor in mathematics at Wesleyan University, to accept a place in the Nautical Almanac Office at Washington, D. C.

E. M. BLAKE, Ph. D. (Columbia, 1893), has been called to Purdue University, Lafayette, Ind., as instructor in mathematics.

G. H. LING, Ph. D. (Columbia, 1896), has been appointed instructor in mathematics at Wesleyan University, Middletown, Conn.

MR. J. J. SKINNER has been promoted to an assistant professorship of mathematics at the Massachusetts Institute of Technology.

DR. DAVIS GARBER, professor of mathematics and astronomy in Muhlenberg College, Allentown, Pa., died on September 22, aged sixty-seven years.

PROFESSOR A. W. PHILLIPS, of Yale University, is engaged in revising the series of mathematical text-books written by the late Professor Loomis.

AT the University of Virginia, Professor W. H. ECHOLS succeeded Professor C. S. VENABLE, who resigned as head of the department of pure mathematics. Dr. J. M. PAGE has been since appointed adjunct professor of mathematics.

A VACANCY was caused in the department of mathematics at Wells College by the resignation of Miss S. A. ACER. It has been filled by the appointment of Miss ANNIE L. MACKINNON, Ph. D. (Cornell). Miss MacKinnon's dissertation, "Concomitant Binary Forms in Terms of the Roots," appeared in the *Annals of Mathematics* for July, 1895.

PROFESSOR E. D. ROE, JR., for the last five years associate professor of mathematics at Oberlin College, has resigned his position in order to spend the next two years in travel and study abroad.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BALL (W. W. R.). Mathematical recreations and problems of past and present times. 3d edition. London, Macmillan, 1896. 8vo. 288 pp. 7s.

BIANCHI (L.). Vorlesungen über Differenzialgeometrie. Autorisierte deutsche Uebersetzung von M. Lukat. (In 2 Lieferungen.) Lieferung I. Leipzig, Teubner, 1896. 8vo. 4 and 336 pp. Mk. 12.00

BINDER (W.). Theorie der unikursalen Plankurven vierter bis dritter Ordnung in synthetischer Behandlung. Leipzig, Teubner, 1896. 8vo. 11 and 396 pp. 2 plates. Mk. 12.00

BEEGGEN (J. VAN DER). Integraalrekening. Versameling van opgeloste Vraagstukken. Zutphen, 1896. 8vo. 4 and 160 pp. 1 plate. Mk. 3.50

COHN (J.). Geschichte des Unendlichkeitsproblems im abendländischen Denken bis Kant. Leipzig, 1896. 8vo. 7 and 261 pp. Mk. 5.00

DEDOFF (T.). Untersuchungen über quadratische Formen. [Diss.] Leipzig, Teubner, 1896. 4to. 40 pp. Mk. 2.80

FISKE (T. S.). The straight line as a minimum length. (*Science*, new series, Vol. IV., 1896, p. 533.)

GOUDIN (J. B.). Considérations élémentaires ou notions sur le calcul infinitésimal et ses dérivés. Paris, 1896. 8vo. Fr. 2.00

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